Journal of Computational Physics **156,** 352–359 (1999)
Article ID jcph.1999.6364, available online at http://www.idealibrary.com on IDEAL®

Numerical Solution of a Generalized Elliptic Partial Differential Eigenvalue Problem

S. R. Otto* and James P. Denier†

**School of Mathematics and Statistics, The University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom; †Department of Applied Mathematics, The University of Adelaide, Adelaide 5005, Australia*
E-mail: S.R.Otto@bham.ac.uk, jdenier@maths.adelaide.edu.au

Received February 18, 1999; revised August 24, 1999

In this article we discuss a method for the solution of non-separable eigenvalue problems. These problems are taken to be elliptic and linear and arise in a whole host of physically interesting problems. The approach exploits finite differences and a pseudo-spectral scheme. We elect to normalise at a single point, which is usually internal to the domain, and exploit the fact that the partial differential equation has not been satisfied at this point to determine whether we have an eigenvalue of the system. The eigenvalue solver is of a local nature and is consequently relatively inexpensive to run. © 1999 Academic Press

Key Words: eigenvalues; elliptic.

1. INTRODUCTION

In order to understand the dynamics of many physical situations, whether they be macro- or microscopical in nature, one is compelled to solve an elliptic eigenvalue problem. By the use of physical arguments, for instance symmetry, it is often possible to reduce these problems to ones governed by ordinary differential equations. However, there are many cases where this is infeasible and any form of *ad hoc* approximation can lead to results that are totally unreliable. Such elliptic eigenvalue problems occur in many different fields, from meteorology [3] and quantum physics [1] to fluid stability [2] and the behaviour of magnetic dynamos [4].

We shall not dwell upon any physical setting which results in such elliptic eigenvalue problems but will merely give a brief description of one which arises in the study of the stability of fluid flows. In the work of Hall and Horseman [2] the authors discuss the susceptability of nonlinear counter-rotating vortices within a boundary layer to secondary instabilities; the existence of such secondary instabilities has been identified as an important factor in the process of transition to turbulence. In a similar vein, the work of Otto, Sarkies, and Denier [8] considers the influence of a spanwise perturbation on the stability

0021-9991/99 $30.00

characteristics of Kelvin–Helmholtz waves within compressible mixing layers. Both articles solve an equation which governs the structure of perturbations to a given basic flow $\overline{u}(y, z)$ in the form of travelling waves proportional to $\exp(i\alpha(x - ct))$. The complex phase speed c is determined as a function of a real streamwise wavenumber, α, by solving

$$\mathcal{L}(P) \equiv \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} - \frac{2}{\overline{u} - c}\frac{\partial \overline{u}}{\partial y}\frac{\partial P}{\partial y} - \frac{2}{\overline{u} - c}\frac{\partial \overline{u}}{\partial z}\frac{\partial P}{\partial z} - \alpha^2 P = 0, \tag{1}$$

subject to suitable boundary conditions (determined by the physics of the particular flow in question). In (1), P is the perturbation pressure, $\overline{u}$ denotes the basic velocity component in the streamwise, x, direction with y and z the ordinates of the other two mutually orthogonal, Cartesian axes. Similar equations arise in a meteorological context with additional terms present to account for buoyancy forcing (a generalisation of the Taylor–Goldstein equation applicable to baroclinic instabilities), and in quantum problems with a potential varying in more than one spatial coordinate (Schrödinger's wave equation). Thus, although our motivation arises from fluid dynamics, there are many other physical problems in which these methods can be used.

In this article we shall detail the numerical scheme used to solve (1) and where possible we shall explain how this approach would be modified to solve problems from different physical settings, in varying geometries. Solutions are given for two model problems from fluid dynamics.

2. FORMULATION

We shall consider the elliptic linear eigenvalue problem (1), for two different fluid stability problems: a periodic array of jets in the neighbourhood of a solid boundary, located at $y = 0$, and a periodic perturbation to an otherwise uniform shear flow. The flows are both taken to be periodic in the spanwise coordinate and as such we shall impose the condition that $P(y, z) = P(y, z + 2\pi)$, so that the perturbation has the same periodicity as the underlying basic flow. In the bounded problem we impose the condition that $\partial P/\partial y = 0$ at $y = 0$ (which is a simple consequence of the impermeability of the plate). In the unbounded problem (and far from the plate in the former problem) we impose the condition that the disturbance pressure decays to zero.

The precise form of the decay conditions is determined by an asymptotic consideration of (1). This is a relatively straightforward exercise for the two model flows considered here, but can become less trivial in other problems. For instance those problems formulated within cylindrical polar coordinates may require the use of Bessel functions, and consequently a knowledge of their asymptotic forms for large arguments. In the present context, the requirement that $|P| \to 0$ at the upper extreme of the region in the bounded problem is imposed at some outer bound $y = y_\infty$ via the Robin condition

$$\frac{\partial P}{\partial y} + \alpha P = 0.$$

In the case of an unbounded domain the decay conditions $|P| \to 0$ as $|y| \to \infty$ are replaced by

$$\frac{\partial P}{\partial y} \pm \alpha P = 0 \qquad \text{at } y = y_{\pm\infty}.$$

In order to discretise (1) we elect to use a pseudo-spectral scheme in the z coordinate and a five point central finite difference scheme in the y coordinate; the computational grid has $N_y \times N_z$ points. The discretized equation (1) can be written as

$$\underline{A}_j \, \underline{P}_{j+2} + \underline{B}_j \, \underline{P}_{j+1} + \underline{C}_j \, \underline{P}_j + \underline{D}_j \, \underline{P}_{j-1} + \underline{E}_j \, \underline{P}_{j-2} = \underline{R}_j, \tag{2}$$

for $j = 2, \ldots, N_y - 1$. Each vector $\underline{P}_j$ is of length N_z and represents the z variation at $y = y_j$. The matrices in (2) are of a size $N_z \times N_z$ and are given by

$$\begin{aligned}
\underline{A}_j &= a_{2,j} \underline{I} + a_{1,j} \operatorname{diag}\big(f_{j1}, \ldots, f_{jN_z}\big), \\
\underline{B}_j &= b_{2,j} \underline{I} + b_{1,j} \operatorname{diag}\big(f_{j1}, \ldots, f_{jN_z}\big), \\
\underline{C}_j &= \big(c_{2,j} - \alpha^2\big) \underline{I} + c_{1,j} \operatorname{diag}\big(f_{j1}, \ldots, f_{jN_z}\big) + \underline{\Delta}^2 + \operatorname{diag}\big(g_{j1}, \ldots, g_{jN_z}\big) \underline{\Delta}, \\
\underline{D}_j &= d_{2,j} \underline{I} + d_{1,j} \operatorname{diag}\big(f_{j1}, \ldots, f_{jN_z}\big), \\
\underline{E}_j &= e_{2,j} \underline{I} + e_{1,j} \operatorname{diag}\big(f_{j1}, \ldots, f_{jN_z}\big),
\end{aligned}$$

together with $\underline{R}_j = \underline{0}$ (where $\underline{I}$ is the $N_z \times N_z$ identity matrix). In these expressions $(a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_{i,j})$ are simply the stencil weights associated with the ith y-derivative at y_j, which can be obtained by using a Taylor series expansion about the grid point in question. The coefficients f_{ij} and g_{ij} are the values of the functions multiplying $\partial P/\partial y$ and $\partial P/\partial z$ in (1) evaluated at the point $y = y_i$ and $z = z_j$. The matrices associated with the z variation are $\underline{\Delta} = \underline{F}^{-1} \underline{D} \, \underline{F}$ and $\underline{\Delta}^2 = \underline{F}^{-1} \underline{D}^2 \underline{F}$, where the matrix $\underline{F}$ transforms into the Fourier space and $\underline{D}$ is the diagonal matrix which multiplies each Fourier component by its mode number and $\sqrt{-1}$ (and hence produces the spatial derivative). These are given by

$$\begin{aligned}
(\underline{F})_{j,k} &= \frac{1}{\sqrt{N_z}} \exp\left(-\frac{2\pi i}{N_z}\left(j - 1 - \frac{N_z}{2}\right)(k-1)\right), \\
(\underline{F}^{-1})_{j,k} &= \frac{1}{\sqrt{N_z}} \exp\left(\frac{2\pi i}{N_z}(j-1)\left(k - 1 - \frac{N_z}{2}\right)\right), \\
\underline{D} &= \operatorname{diag}\left(-\frac{N_z}{2} i, \ldots, \left(\frac{N_z}{2} - 1\right) i\right).
\end{aligned}$$

Notice at the points near the boundaries (in y) biased stencils are used and hence $\underline{E}_2$ and $\underline{A}_{N_y-1}$ are empty. The boundary conditions constitute the first and last equations: for the semi-infinite (wall) bounded problem we have

$$\begin{aligned}
j &= 1 \qquad && a_{1,1} \underline{P}_3 + b_{1,1} \underline{P}_2 + c_{1,1} \underline{P}_1 = \underline{0} \\
j &= N_y && \big(c_{1,N_y} + \alpha\big) \underline{P}_{N_y} + d_{1,N_y} \underline{P}_{N_y-1} + e_{1,N_y} \underline{P}_{N_y-2} = \underline{0}
\end{aligned} \tag{3a}$$

while for the infinite (unbounded) case we require

$$\begin{aligned}
j &= 1 \qquad && a_{1,1} \underline{P}_3 + b_{1,1} \underline{P}_2 + (c_{1,1} - \alpha) \underline{P}_1 = \underline{0} \\
j &= N_y && \big(c_{1,N_y} + \alpha\big) \underline{P}_{N_y} + d_{1,N_y} \underline{P}_{N_y-1} + e_{1,N_y} \underline{P}_{N_y-2} = \underline{0}.
\end{aligned} \tag{3b}$$

Notice that it is possible to cater for boundary conditions of the form $k(z)\partial P/\partial y + l(z)P = 0$, in which case the scalars in (3) are replaced by the matrices which will arise from the spectral decomposition of the functions $k(z)$ and $l(z)$. The resulting system is now block penta-diagonal, which can be solved by Gaussian elimination in a manner which minimises the storage requirements. The question now arises as to how to determine the complex phase speed c. In general direct methods are prohibitively expensive in terms of CPU time therefore an iterative method is the natural choice. In order to implement such an iterative scheme it is necessary to normalise the system (2) in some suitable manner. In bounded problems, for instance those discussed in Hall and Horseman [2] and Otto [5], a renormalisation was used, in which the boundary condition at $y = 0$ was replaced with $\partial P/\partial y|_{y=0} = f(z)$. Iteration on the eigenvalue c was used until the integral of P^2 at $y = 0$ becomes large; when this integral becomes suitably large the boundary condition $\partial P/\partial y|_{y=0} = 0$ is satisfied following a renormalisation. In unbounded problems it is possible to use an alternating Schwarz technique.

Here, and in the work of Otto *et al.* [8], we normalise at a single internal point rather than at a boundary. We choose a point $(y, z)_{normalise}$ (notice that this point can be at a boundary, provided the boundary conditions are not of a Dirichlet form) and replace the discretised equation at the normalising point by

$$P\Big|_{(y,z)_{normalise}} = 1.$$

The fact that the differential equation has not been satisfied at this point is used to iterate on the eigenvalue until

$$\Big\|\mathcal{L}(P)\Big|_{(y,z)_{normalise}}\Big\| < \text{tolerance},$$

where $\| \bullet \|$ is some suitable norm. We note that $\mathcal{L}(P)|_{(y,z)_{normalise}}$ is a function of c, and as such can be used to determine eigenvalues of the system. If the normalising point is on a boundary then the fact that the boundary condition has not been satisfied can be used for the iteration process. A conventional secant method is adequate for the iterative procedure. This technique has the advantage over that employed by Hall and Horseman [2] in that the boundary conditions are satisfied explicitly and there is no reliance on choosing a function $f(z)$ (as there would be in an alternating Schwarz technique). The rate of convergence onto the eigenvalues is much improved and calculations for different values of N_z are easier to compare. It is possible to do inexpensive parameter studies using relatively small values of N_z; larger values of N_z can then be used to further resolve any important, or interesting, results arising from such a parameter study.

Runs for moderate grid sizes ($N_y = 100 - 400$ and $N_z = 16 - 64$) were performed on a Silicon Graphics Workstation R5000; however, the larger parameter runs were performed on a SGI Power Challenge and a Fujitsu VPP300. We shall now discuss two model problems arising in fluid dynamics which serve to demonstrate the utility of the algorithm described above.

3. RESULTS

We shall initially consider a model of the flow within a periodic array of jets above a plane boundary. The "basic flow" is taken to be a modified form of the asymptotic suction profile, namely

$$u_b(y) = 1 - e^{-y}(1 + \zeta y),$$

where the parameter ζ is used to introduce an inflection point into the flow ($\zeta = 0$ corresponds to the conventional asymptotic suction profile). Note that the flow u_b has an inflection point at $y = (2\zeta - 1)/\zeta$ (provided $\zeta > 1/2$). Such inflectional flows will support growing disturbances in the form of inviscid travelling waves. It was shown in [5] that spanwise periodic perturbations (Görtler vortices) will increase the instability of an inviscidly unstable flow (in this case a boundary layer with an adverse pressure gradient). It is also possible for non-inflectional profiles to be modified by the spanwise perturbations to become inviscidly unstable, such as those considered by Hall and Horseman [2] and Otto and Denier [6].

The total flow is taken to be

$$\overline{u}(y, z) = u_b(y) + \frac{\Delta}{2}\exp\left(-(y - y_c)^2\right)(\cos z + 1)u_b(y)$$

for the case of the periodic jet array, where the variable y_c denotes the centre of the "jets." The second flow of interest is taken to be

$$\overline{u}(y, z) = y + \frac{\Delta}{2}\exp\left(-(y - y_c)^2\right)(\cos z + 1). \tag{4}$$

(In this case the Robin boundary condition must be modified in order that, in line with the large y asymptotic behaviour of the solutions of (1), $P \to ye^{\mp\alpha y}$ as $y \to \pm\infty$.)

In Fig. 1 we present a plot of the normal perturbation (real and imaginary part) for a value of $\alpha = 0.2$; the corresponding eigenvalue is $c = 0.232 + 0.527i$ and was calculated using $N_y = 100$ and $N_z = 32$. In Fig. 2 we present in part (a) a plot of the imaginary part of the

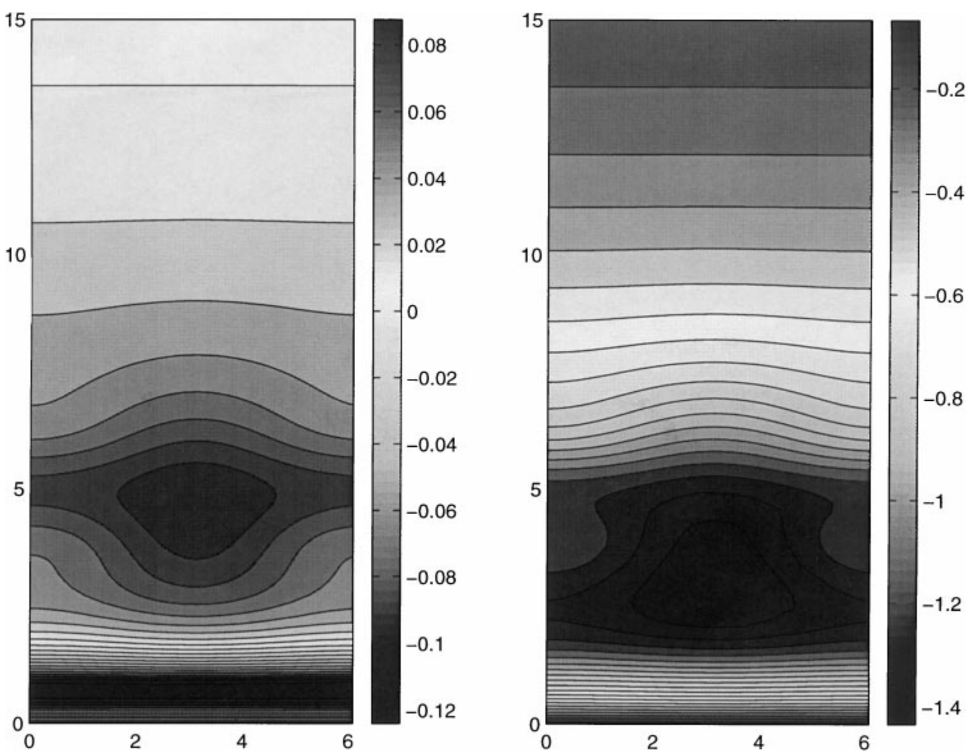
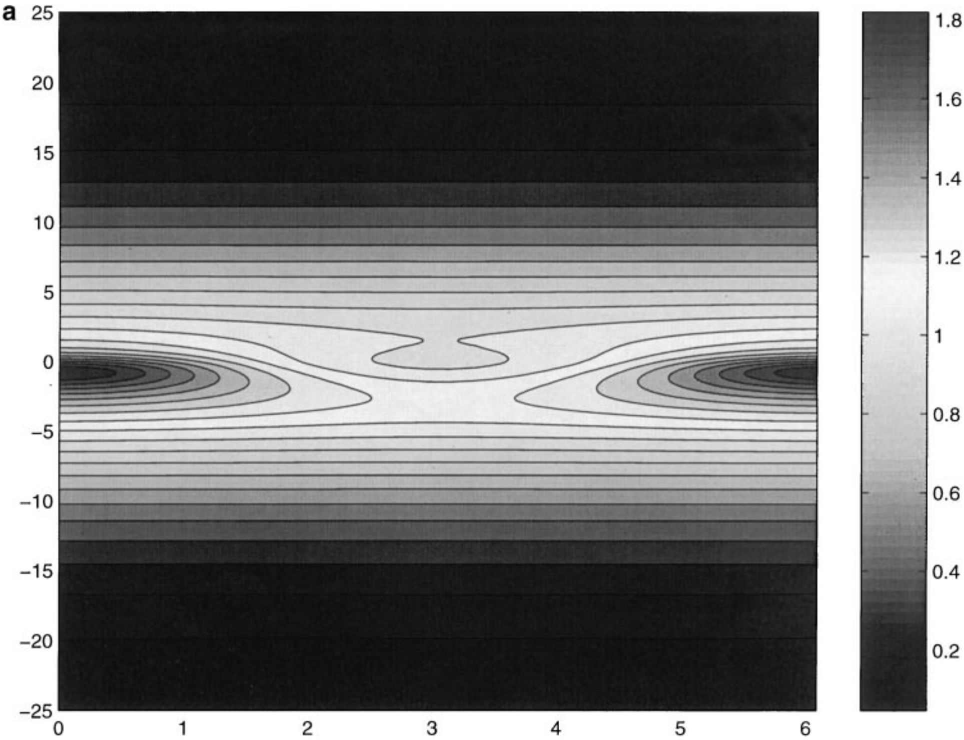


FIG. 1. The normal component of the velocity perturbation for the case (real and imaginary parts) $y_c = 5$, $\zeta = 1$, and $\alpha = 1/5$, corresponding to $c = 0.232 + 0.527i$ ($N_y = 100$, $N_z = 32$).



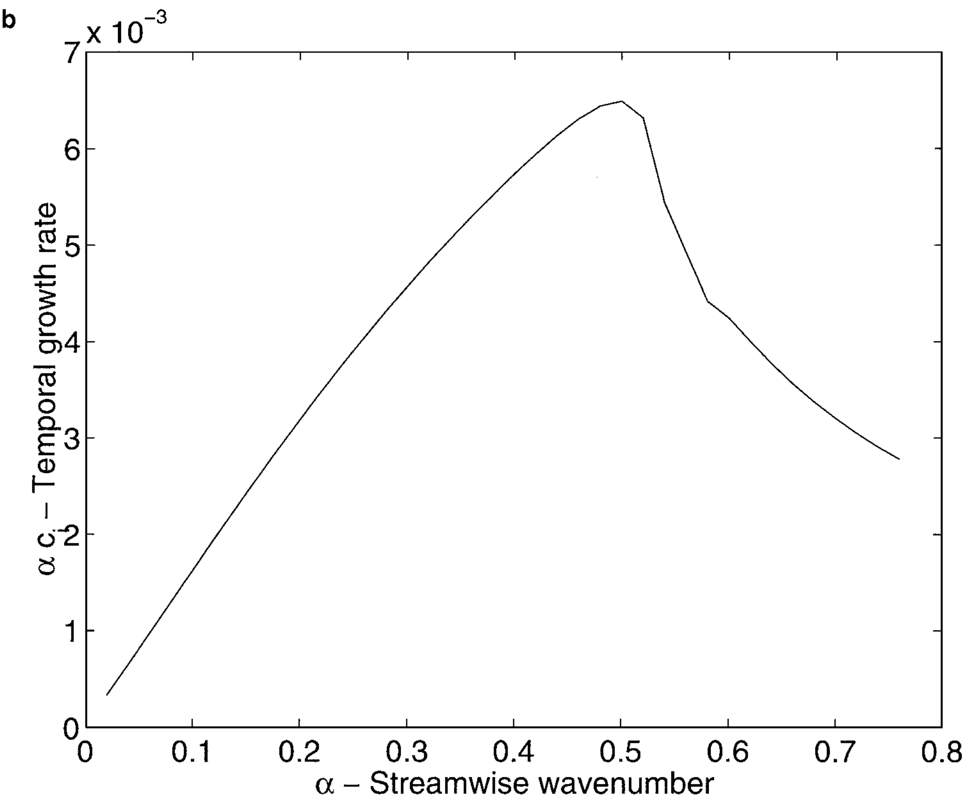


FIG. 2. The effect of a spanwise perturbation to an unbounded uniform shear, showing the imaginary part of the pressure (a). The parameters used are $\alpha = 1/5$ and $y_c = 0$ with a vortex amplitude of 1.5, which yields $c = -.0209 + 0.01593i$ ($N_y = 400$, $N_z = 32$). The corresponding growth rate is shown in (b).

eigenfunction, for a value of $\alpha = 0.2$ with $y_c = 0$ and in part (b) a plot of the growth rate αc_i versus α; the base flow $\bar{u}$ is given by (4). Note here that the "smallness" of the growth rate is due to the fact that the instability is due entirely to the introduction of the inflectional two-dimensional component to the basic shear flow $u_b = y$. Both flows are unstable and the eigenvalues and corresponding eigenfunctions are readily computed. To produce the results in Fig. 2b (which represents 45 data points with a computational grid 400 by 32) took 4900 s on a Silicon Graphics O2 (R5000).

4. CONCLUSIONS

We have described a method whereby eigenvalue problems governed by partial differential equations can be solved efficiently and cheaply (in terms of CPU). Here a normalisation condition is imposed at an internal point and any boundary conditions are satisfied explicitly. The resulting inhomogeneous system can then be inverted (efficiently in the case of the banded systems encountered in the example considered here) and iteration upon the eigenvalue carried out until the discretized equation is satisfied (to within some prescribed tolerance) at the normalisation point.

The utility of this technique is threefold. First, it provides an efficient (local) method whereby eigenvalue problems governed by partial differential equations can be solved. The normalisation procedure is a natural one which is easy to implement. Second, there is no restriction on the form of the partial differential equations which can be tackled using this technique. It can be as readily applied to the single equation discussed here as it can to higher order systems such as those encountered in studies of viscous stability theory or more complex flows, as in Otto and Streett [7], as well as to other forms of differential operators such as the ballooning Schrödinger equation arising in plasma physics [1]. Importantly, from the standpoint of tackling "real world" problems the differential operators are not required to be separable and the domain of the resulting eigenfunction can be bounded, semi-infinite, or unbounded (in one or more of the independent variables). In the case of unbounded domains, the only modifications that are required to the scheme result from the form of boundary conditions that act to close the system. Thus, for example, systems in which the domain of the "flow" $u(y, z)$ is infinite in the z-direction (with perhaps compact support in z) can be accommodated provided the correct form of decay boundary conditions are implemented. This technique can also be employed to solve eigenvalue problems which possess removable singularities at a boundary. Many such problems occur within fluid dynamics when considering the stability of flows best described by using a cylindrical polar coordinate system. The choice of normalisation away from the vicinity of the singularity, together with the ability to explicitly satisfy boundary conditions, makes this method particularly useful in such cases.

Finally, it is worth emphasizing that the algorithm described above is equally suited to solving spatial eigenvalue problems governed by elliptic partial differential operators. In this case one fixes the real frequency αc and iterates on the complex "wavenumber" α until the convergence criteria is satisfied.

REFERENCES

1. R. L. Dewar, Spectrum of the ballooning Schrödinger equation, *Plasma Phys. Control Fusion* **39**, 453 (1997).
2. P. Hall and N. J. Horseman, The linear inviscid secondary instability of longitudinal vortex structures in boundary layers, *J. Fluid Mech.* **232**, 357 (1991).

3. M. E. McIntyre, On the non-separable baroclinic parallel flow instability problem, *J. Fluid Mech.* **40**, 273 (1970).

4. K. H. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge Univ. Press, Cambridge, UK, 1978).

5. S. R. Otto, On the secondary instability of Görtler vortices in three-dimensional boundary layers, in *Proceedings of the IUTAM Symposium on Nonlinear Instability and Transition in Three-Dimensional Boundary Layers,* edited by P. W. Duck and P. Hall (Kluwer Academic, Dordrecht, 1995), p. 85.

6. S. R. Otto and J. P. Denier, On the secondary instability of the most dangerous Görtler vortex, in *IUTAM Symposium on Nonlinear Instability of Nonparallel Flows*, edited by S. P. Lin, W. R. C. Phillips, and D. T. Valentine (Springer-Verlag, New York/Berlin, 1993), p. 290.

7. S. R. Otto and C. L. Streett, On the secondary instability of swept wing flows, in *Proceedings of the ICASE Workshop on Transition, Turbulence and Combustion,* edited by M. Y. Hussaini, T. B. Gatski, and T. L. Jackson (Kluwer Academic, Dordrecht, 1994), p. 329.

8. S. R. Otto, J. M. Sarkies, and J. P. Denier, On the effect of Görtler vortices on Kelvin–Helmholtz modes, in preparation.